A Discrete Equilibrium (Sperner's Lemma Revisited)

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Abstract

We provide a purely combinational proof of Sperner's lemma without resorting to simplicial subdivisions. Given two natural integers N and n, we consider the set $\Delta(N, n)$ of vectors $x = \frac{1}{N}(k_1, k_2, \ldots, k_n)$ where the k_i are natural integers between 0 and N whose sum is N. A *cluster* in $\Delta(N, n)$ is a subset $K = \{x^1, x^2, \ldots, x^n\}$ of n vectors where two consecutive vectors x^k and x^{k+1} differ only by $1/N$ in two consecutive coordinates. Let $\{A_1, A_2, \ldots, A_n\}$ be a covering of $\Delta(N, n)$ such that $A_i \subset \{x \in \Delta(N, n) : x_i > 0\}$. A *discrete equilibrium* is a cluster K that meets every set A_i. The existence of a discrete equilibrium is a revision of Sperner's Lemma. By making $N \to \infty$, we unsurprisingly find the KKM Lemma and the fixed point theorems.